

A UNIQUE POLAR REPRESENTATION OF THE HYPERANALYTIC SIGNAL

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ABSTRACT

The hyperanalytic signal is the straight forward generalization of the classical analytic signal. It is defined by a complexification of two canonical complex signals, which can be considered as an inverse operation of the Cayley-Dickson form of the quaternion. Inspired by the polar form of an analytic signal where the real instantaneous envelope and phase can be determined, this paper presents a novel method to generate a polar representation of the hyperanalytic signal, in which the continuously complex envelope and phase can be uniquely defined. Comparing to other existing methods, the proposed polar representation does not have sign ambiguity between the envelope and the phase, which makes the definition of the instantaneous complex frequency possible.

Index Terms— hyperanalytic signal, quaternionic signal, polar representation, instantaneous complex envelope, instantaneous complex frequency

1. INTRODUCTION

In the signal processing community, the analytic signal is computed as a complexification of a real-valued signal, generated by the signal itself and its Hilbert transform, see, e.g., [1]. This is a well-known model for signal characterization. Based on it, the instantaneous amplitude or envelope, the instantaneous phase, and, thus, the instantaneous frequency of the given signal can be well identified. Then, with the obtained time and instantaneous amplitude and frequency quantities, the constructed time-frequency-amplitude (TFA) spectrum may illustrate valuable information for data classification, signal decomposition and many other applications.

Nowadays, in many applications, e.g., geophysical [2, 3] or meteorological [4] data analysis, signals simultaneously sampled from multiple sensors may not be efficiently characterized based on the classical model. Moreover, the sub-components of non-stationary multivariate signals by using an adaptive data decomposition method, e.g., [4, 5], also need a versatile TFA representation. Therefore, it is necessary to develop a solid theory for multivariate signal analysis.

A multivariate version based on quaternions, so-called monogenic signals, with applications to images was developed in [6]. In [7], the concept of hyperanalytic signal (H-signal) was proposed to provide a hypercomplex representation of the given complex or bivariate signal by using a one-sided quaternionic Fourier transform [8]. Inspired by the Cayley-Dickson form, the generated H-signal may be represented in a polar form where the corresponding envelope and the phase are complex [9]. However, there is a sign ambiguity between the envelope and the phase, which results in the fact that the definition of the instantaneous complex frequency is unclear. [2] introduced another method for complex signal characterization based on the modulated elliptical model [10].

This paper firstly reviews the quaternion computation and the H-signal construction from a given complex signal. Then the polar representation of the quaternion and the corresponding sign ambiguity will be explained. After that we present a novel envelope recovery algorithm based on a linear zero-crossing prediction, which results in a unique polar form for the H-signal. Thus, the instantaneous complex frequency can be naturally defined. We illustrate the efficiency of the proposed method via a representative numerical study, and close with a discussion and some final remarks.

2. QUATERNIONIC SIGNALS

2.1. Quaternion and its operations

A quaternion, introduced by Hamilton in 1843, is for the Cartesian coordinate axes the set $\mathbb{H} := \{q_r + iq_i + jq_j + kq_k : q_r, q_i, q_j, q_k \in \mathbb{R}\}$ where $\{1, i, j, k\}$ customarily denotes the basis. Every element $q \in \mathbb{H}$ can be uniquely written in a linear combination of these basis elements.

The real part of q , denoted as the scalar, is $\mathcal{S}(q) := q_r$, while the residual is denoted by $\mathcal{V}(q) := q - \mathcal{S}(q)$. q is called a pure quaternion when $q_r = 0$. Each basis element is considered as the root of -1 . Multiplications among them satisfy $i^2 = j^2 = k^2 = ijk = -1$, which results in the potential rules $ij = -ji = k, jk = -kj = i, ki = -ik = j$. Note that the quaternion multiplication is not commutative, e.g., $qp \neq pq$, for $q, p \in \mathbb{H}$. The conjugate of q is defined as

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$\bar{q} = q_r - iq_i - jq_j - kq_k$, and, thus, the Euclidean norm of q is $\|q\| := \sqrt{q\bar{q}} = \sqrt{q_r^2 + q_i^2 + q_j^2 + q_k^2}$. Then the inverse of q is given by $q^{-1} := \frac{\bar{q}}{\|q\|^2}$. By applying the Cayley-Dickson form, any quaternion can be represented as a pair of complex numbers, e.g. $q = q_r + iq_i + jq_j + kq_k = (q_r + iq_i) + (q_j + iq_k)j =: z_1 + z_2j$, for $z_1, z_2 \in \mathbb{C}$.

Definition 2.1 Given a quaternion $q \in \mathbb{H}$, the exponential and the natural logarithm of q can be defined by

$$e^q := e^{\mathcal{S}(q)} \left(\cos(\|\mathcal{V}(q)\|) + \frac{\mathcal{V}(q)}{\|\mathcal{V}(q)\|} \sin(\|\mathcal{V}(q)\|) \right), \quad (1)$$

$$\ln(q) := \ln(\|q\|) + \frac{\mathcal{V}(q)}{\|\mathcal{V}(q)\|} \arccos\left(\frac{\mathcal{S}(q)}{\|q\|}\right). \quad (2)$$

2.2. Hyperanalytic signal

For any complex signal $z(t) \in \mathbb{C}$, the quaternionic Fourier transform (QFT) of $z(t)$ may have left, right and double-sided versions, since the exponential kernel placed in different positions leads to different results. In the present context, the right QFT is appropriate for the H-signal construction [7, 8].

Definition 2.2 Given a complex signal $z(t) \in \mathbb{C}$, and a unit quaternion $\mu \in \mathbb{H}$, the right QFT of $z(t)$ with respect to (w.r.t) the μ -axis is defined by

$$\hat{z}_\mu(\omega) = \mathcal{F}_\mu^q[z(t)] := \int_{\mathbb{R}} z(t) e^{-\mu\omega t} dt \quad (3)$$

If we replace the μ -axis with a canonical j -axis, the \mathcal{F}_j^q of a real signal $a(t)$ can be considered as the Fourier transform of $a(t)$. Thus, the QFT can be sped up by the fast Fourier transform (FFT) as follows.

Corollary 2.3 Given a complex signal $z(t) = z_r(t) + iz_i(t)$, $t, z_r(t), z_i(t) \in \mathbb{R}$, and the quaternionic j -axis, the right QFT of $z(t)$ can be expressed in terms of the Fourier transform

$$\hat{z}_j(\omega) = \mathcal{F}_j^q[z(t)] = \mathcal{F}_j[z_r(t)] + i\mathcal{F}_j[z_i(t)] \quad (4)$$

Based on the QFT, we can modify the Hilbert transform (HT) in the Hamilton space. In the time domain, the output of the HT of a complex signal should still be a complex signal that is orthogonal to the input. Then this pair of signals can be combined in a Cayley-Dickson form to generate a quaternionic signal (Q-signal). In the frequency domain, the frequency of such generated Q-signal should be physically meaningful, i.e., nonnegative.

Definition 2.4 Given a complex signal $z(t) \in \mathbb{C}$ and a unit quaternion $\mu \in \mathbb{H}$, the quaternionic Hilbert transform (QHT) of $z(t)$ w.r.t the μ -axis is defined by

$$\mathcal{H}_\mu^q[z(t)] := \mathcal{F}_\mu^{q-1} [-\mu \operatorname{sgn}(\omega) \mathcal{F}_\mu^q[z(t)]] \quad (5)$$

Here, \mathcal{F}_μ^{q-1} means the inverse QFT. The QHT can also be defined in the time domain by $\mathcal{H}_\mu^q[z(t)] := \operatorname{PV}(z(t) * \frac{1}{\pi t})$, where PV denotes the Cauchy principal value and $*$ represents the convolution. Replacing again the μ -axis with the j -axis, (5) can be further simplified.

Corollary 2.5 Given a complex signal $z(t) = z_r(t) + iz_i(t)$, $t, z_r(t), z_i(t) \in \mathbb{R}$, and the quaternionic j -axis, the QHT of $z(t)$ can be expressed in terms of the HT

$$\mathcal{H}_j^q[z(t)] = \mathcal{H}[z_r(t)] + i\mathcal{H}[z_i(t)]. \quad (6)$$

Similar to the analytic signal model, we can construct the H-signal for any given complex signal, which is indeed a subset of the Q-signal.

Definition 2.6 Given a complex signal $z(t) \in \mathbb{C}$, the hyperanalytic signal is defined by

$$s(t) := z(t) + o(t)j = z(t) + \mathcal{H}_j^q[z(t)]j, \quad (7)$$

where $o(t)$ is the QHT of $z(t)$ w.r.t the j -axis.

3. HYPERANALYTICAL SIGNAL MODEL

3.1. Sign ambiguity in the polar form

Suppose the quaternion $q = q_r + iq_i + jq_j + kq_k$ is given, $q_r, q_i, q_j, q_k \in \mathbb{R}$, and its polar representation is in the form of $q := Ae^{Bj}$, where $A := a + ib$, $B := c + id$, and $a, b, c, d \in \mathbb{R}$. As Bj is a pure quaternion, according to (1), the exponential of Bj can be expressed as

$$\begin{aligned} e^{Bj} &:= \alpha + j\beta + k\gamma \\ &:= \cos(\|B\|) + j\frac{c}{\|B\|} \sin(\|B\|) + k\frac{d}{\|B\|} \sin(\|B\|), \end{aligned} \quad (8)$$

where $\|B\| = \sqrt{c^2 + d^2}$ [9]. Then, we arrive at the equations

$$\begin{aligned} q &= q_r + iq_i + jq_j + kq_k := Ae^{Bj} \\ &= a\alpha + ib\alpha + j(a\beta - b\gamma) + k(a\gamma + b\beta). \end{aligned} \quad (9)$$

Since the complex envelope A can be expressed in polar form by $A := \|A\|e^{i\phi_A} = \|q\|e^{i\phi_A}$, we can determine that the axis of the known complex component $q_r + iq_i$ equals to the axis of $a\alpha + ib\alpha$. In other words, with an axis operator defined as $\mathcal{A}(a + ib) := \frac{a + ib}{\|a + ib\|}$, we have following relationship

$$e^{i\phi_A} = \mathcal{A}(a + ib) = \frac{A(q_r + iq_i)}{\operatorname{sgn}(\alpha)}, \quad (10)$$

where $\operatorname{sgn}(\cdot)$ is the signum function. Obviously, this leads to an ambiguity in sign between the complex envelope A and the phase B since the $\operatorname{sgn}(\alpha)$ is unknown for computing $e^{i\phi_A}$.

3.2. Complex envelope recovery

To simplify the polar representation of the H-signal, denoted by $q(t) = A(t)e^{B(t)j}$, $t \in [0, T]$, we assume a unit signal to be processed in this section, i.e., $\|q(t)\| = 1$. Then the unwanted $\operatorname{sgn}(\alpha)$ in (10) can be removed by taking the modulus of the real and imaginary components on both sides,

$$\begin{aligned} |\cos(\phi_A(t))| &= |a(t)| = |\tilde{q}_r(t)|, \\ |\sin(\phi_A(t))| &= |b(t)| = |\tilde{q}_i(t)|, \end{aligned} \quad (11)$$

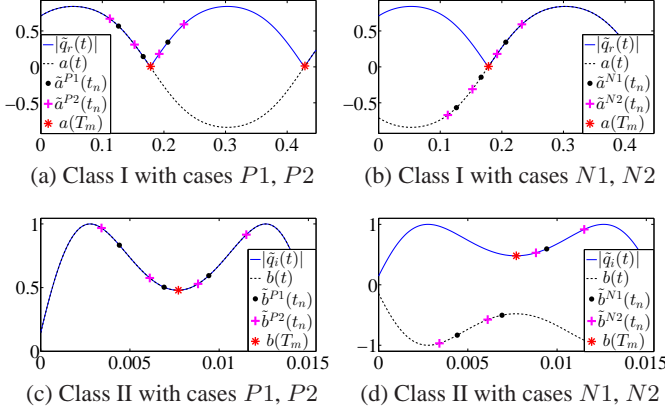


Fig. 1. Eight possibilities for the sign recovery of the envelope $A(t) = a(t) + ib(t)$ based on the modulus components of the axis $\mathcal{A}(q_r(t) + iq_i(t))$. P (positive) and N (negative) denote the sign of the former half-period of the recovered component $\tilde{a}(t)$ and $\tilde{b}(t)$. Black dots and magenta pluses denote sampling cases 1 and 2, and red stars imply the ideal local minima (Color online).

where $\mathcal{A}(q_r(t) + iq_i(t)) := \tilde{q}_r(t) + i\tilde{q}_i(t)$.

Recall that we are considering a continuous hyperanalytical signal model, in which the complex envelope $A(t)$ and, thus, the real phase $\phi_A(t)$ should be continuous. Therefore, if the initial range of the phase was limited, i.e., $\phi_A(0) \in [0, \frac{\pi}{2}]$, in view of the continuity, the recovered envelopes $\tilde{a}(t)$ and $\tilde{b}(t)$ could be determined independently, as $|\tilde{q}_r(t)|$ and $|\tilde{q}_i(t)|$ are already known. In detail, since the sign changing of the envelope $a(t)$ or $b(t)$ only occurs at zero-crossing (ZC) position which is nothing but the local minimum of the modulus $|\tilde{q}_r(t)|$ or $|\tilde{q}_i(t)|$, in principle, we can recover the envelope by retrieving the sign of every half-period (HP) of the modulus signal from beginning to the end, where the HP is defined as the interval between every two nearest local minima of the modulus signal.

However, the local minimum of the modulus signal may be positive but not the ZC because we do not require that the phase $\phi_A(t)$ is monotonically non-decreasing. Therefore, we need to classify all cases into two classes: class I denotes the case the local minimum is the ZC, while class II implies a positive local minimum. In addition, for discrete data, the accuracy of the local minimum position is affected by the sampling rate, which means that the current local minimum may be the last point of the former HP (case 1), or the first point of the following one (case 2). Therefore, if we ignore special cases for stationary points, in total, there will be eight possibilities which may occur around the local minimum. For instance, in class I, we have to consider the former HP is positive (case P) or negative (case N), each of which contains another two sampling cases. Fig. 1 gives a comprehensive illustration of all possibilities.

To distinguish these different possibilities, we employ

a linear ZC prediction method based on every two successive samples around the local minimum. Take the case $P1$ in Fig. 1 (a) as an example. Black dots are denoted by $\tilde{a}(t_{n-1})$, $\tilde{a}(t_n)$ and $\tilde{a}(t_{n+1})$, among which t_n corresponds to the local minimum. The predicted ZC position is

$$T_{\tilde{a}}^n := t_{n+1} - \tilde{a}(t_{n+1}) \frac{t_{n+1} - t_n}{\tilde{a}(t_{n+1}) - \tilde{a}(t_n)}. \quad (12)$$

Similarly, $T_{\tilde{b}}^n$ can be calculated for the imaginary component. Then, with the information of the sign of the samples $\tilde{a}(t_{n-1})$ and $\tilde{a}(t_{n+1})$ and the estimates $T_{\tilde{a}}^n$ and $T_{\tilde{a}}^{n-1}$, we can determine to which case the current local minimum belongs. Also considering the black dots in Fig. 1 (a), we can firstly determine that the former HP is positive (case P) as $\text{sgn}(\tilde{a}(t_{n-1})) = \text{sgn}(\tilde{a}(t_{n+1}))$. Secondly, we can determine the class I as the estimate $T_{\tilde{a}}^{n-1}$ is valid ($t_n \leq T_{\tilde{a}}^{n-1} \leq t_{n+1}$), and simultaneously the sampling case 1 as the estimate $T_{\tilde{a}}^n$ is invalid ($T_{\tilde{a}}^n < t_{n-1}$). Therefore, we can keep the sign of $\tilde{a}(t_n)$ and then change the sign of the following NP from t_{n+1} to the former point of the next local minimum.

Since the case determination contains many IF-ELSE conditions, we only present a simplified envelope recovery algorithm as follows. The accuracy of the ZC prediction is guaranteed as the sampling frequency is high enough, otherwise the instantaneous frequency cannot be correctly estimated because of the violation of the sampling theorem.

Algorithm : Complex Envelope Recovery Algorithm

1. Initialize the recovered components $\tilde{a}(t) = |\tilde{q}_r(t)|$ ($\tilde{b}(t) = |\tilde{q}_i(t)|$), and the range of $\phi_A(0)$, e.g., $\phi_A(0) \in [0, \frac{\pi}{2}]$;
 2. Detect the local minimum of $\tilde{a}(t)$ ($\tilde{b}(t)$), and retrieve the sign of the first HP based on the range of the phase;
 3. Recover the envelope by retrieving the sign of the flowing HP based on the sign of the former HP and the determined case at the local minimum $\tilde{a}(t_n)$ ($\tilde{b}(t_n)$);
 4. Output the complex envelope $A(t) = \tilde{a}(t) + i\tilde{b}(t)$.
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3.3. Unique polar representation

Once the complex envelope is recovered, the quaternionic carrier $e^{B(t)j}$ can be computed by

$$e^{B(t)j} := \alpha(t) + j\beta(t) + k\gamma(t) = \frac{\bar{A}(t)q(t)}{\|q(t)\|^2}, \quad (13)$$

and the complex phase $B(t)$ can be derived based on (2)

$$B(t) := c(t) + id(t) = \mathcal{A}(\beta(t) + i\gamma(t)) \arccos(\alpha(t)). \quad (14)$$

From (8), we know that there is still a sign ambiguity between $\sin(\|B(t)\|)$ and $c(t)$ or $d(t)$. However, since both $\sin(\cdot)$ and $\cos(\cdot)$ functions are periodic, we have that $\sin(\|B(t)\|) = \sin(\|B(t)\| \pm 2m\pi)$, and $\cos(\|B(t)\|) = \cos(\|B(t)\| \pm 2m\pi)$, $m \in \mathbb{N}$. Thus, it is reasonable to

assume that $c(t), d(t)$ are non-negative and monotonically non-decreasing, and the initial phase $c(0), d(0)$ should satisfy $\|B(0)\| := \sqrt{c(0)^2 + d(0)^2} \in [0, 2\pi)$. Finally, we can uniquely retrieve the phases $c(t)$ and $d(t)$ based on the unwrapped $\arccos(\alpha(t))$ in (14). The reason to unwrap the phase $\arccos(\alpha(t))$ but not the ones $\check{c}(t)$ and $\check{d}(t)$ (which are directly computed in (14) without unwrapping) is because only the phase $\arccos(\alpha(t))$ has a fixed period 2π . Then the retrieved phases $\tilde{c}(t)$ and $\tilde{d}(t)$ can be considered as approximations of the ideal ones that are monotonically non-decreasing. Thus, we have proved the following result.

Theorem 3.1 *Given a complex signal $z(t) \in \mathbb{C}$, the hyperanalytic signal can be constructed by $s(t) := z(t) + \mathcal{H}_j^q[z(t)]j$, $s(t) \in \mathbb{H}$, which has a unique polar form $s(t) = A(t)e^{B(t)j}$, $A(t), B(t) \in \mathbb{C}$, if $(A(t), B(t))$ is the canonical complex pair where $A(t) := \|s(t)\|e^{i\phi_A(t)}$, $\phi_A(0) \in [0, \frac{\pi}{2}]$, and $B(t) := c(t) + id(t)$, $c(t), d(t) \geq 0$, $\|B(0)\| \in [0, 2\pi)$.*

Bearing in mind that the instantaneous frequency should be nonnegative, therefore, each component of the unwrapped complex phase $B(t)$ should be monotonically non-decreasing, and the unwrapped phase $\phi_A(t)$ is the same if the complex envelope $A(t)$ is an analytic signal.

Definition 3.2 *Given a complex signal $z(t) \in \mathbb{C}$ and let the polar form of its hyperanalytic signal be $s(t) := A(t)e^{B(t)j} := \|s(t)\|e^{i\phi_A(t)}e^{(c(t)+id(t))j}$, $\phi_A(t), c(t), d(t) \geq 0$. The instantaneous complex frequency of $z(t)$ is defined by*

$$f_B(t) := f_{B_r}(t) + if_{B_i}(t) = \frac{1}{2\pi} \left(\frac{d(c(t))}{dt} + i \frac{d(d(t))}{dt} \right), \quad (15)$$

and the instantaneous frequency of the complex envelope $A(t)$ is defined by $f_A(t) := \frac{d(\phi_A(t))}{2\pi dt}$ if $A(t)$ is an analytic signal.

4. NUMERICAL STUDY

To illustrate the efficiency of the proposed method, we design a representative hyperanalytical signal model

$$s(t) := e^{-t} e^{7 \sin(2\pi t) i} e^{(40\pi t + i(20\pi t + 4 \cos(2\pi t))) j}, \quad (16)$$

for $t \in [0, 0.4]$, from which one can define $A(t) := a(t) + ib(t) = e^{-t} e^{7 \sin(2\pi t) i}$, and $B(t) := c(t) + id(t) = 40\pi t + i(20\pi t + 4 \cos(2\pi t))$. Thus we can obtain the given complex signal $z(t) = z_r(t) + iz_i(t)$ based on (7) and (9), and determine the instantaneous complex frequency by $f_B(t) := f_{B_r}(t) + if_{B_i}(t) = 20 + i(10 - 4 \sin(2\pi t))$. Here, $f_A(t)$ is not well defined since it can be negative.

Fig. 2 illustrates all respective results for the given signal $z(t)$. In sub-figures (a) and (b), $\tilde{A}(t) := \tilde{a}(t) + i\tilde{b}(t) = \|q(t)\| \mathcal{A}(q_r(t) + iq_i(t))$ is the reconstructed envelope based on (9), which contains the sign ambiguity. Obviously, the recovered $\tilde{A}(t) := \tilde{a}(t) + i\tilde{b}(t)$ using the proposed method coincides strongly with the ideal complex envelope. Sub-figures

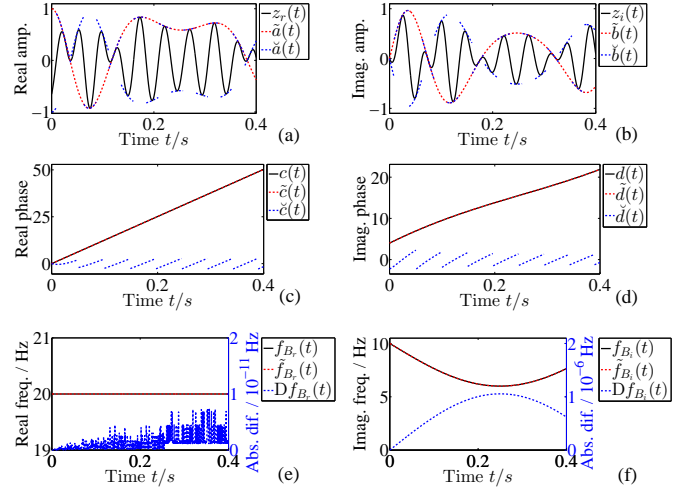


Fig. 2. Numerical results for the H-signal model. Top: real and imaginary (R&I) parts of the given complex signal $z(t)$, the recovered envelope $\tilde{A}(t)$, and the one $\check{A}(t)$ containing the sign ambiguity; Middle: R&I parts of the ideal complex phase $B(t)$, the recovered one with $(\tilde{B}(t) := \tilde{c}(t) + i\tilde{d}(t))$ and without $(\check{B}(t) := \check{c}(t) + i\check{d}(t))$ unwrapping; Bottom: R&I parts of the ideal instantaneous complex frequency $f_B(t)$, the estimated one $\tilde{f}_B(t)$ based on $\tilde{B}(t)$, and the absolute difference between each of them, $Df_{B_r}(t) := |f_{B_r}(t) - \tilde{f}_{B_r}(t)|$, $Df_{B_i}(t) := |f_{B_i}(t) - \tilde{f}_{B_i}(t)|$ (Color online).

(c) and (d) imply the importance of the phase unwrapping, while sub-figures (e) and (f) show the efficiency of the estimation of the instantaneous complex frequency. Since the real component of $f_B(t)$ is a constant, the absolute difference between it and the estimated one \tilde{f}_{B_r} is around machine accuracy. However, the absolute difference between f_{B_i} and \tilde{f}_{B_i} is larger since f_{B_i} is nonlinear and thus the corresponding estimation accuracy is corrupted by the discrete derivative computation at different time positions.

5. CONCLUSION

We presented an efficient method for the unique polar representation of the hyperanalytic signal that is constructed from any given complex signal with continuous real and imaginary components. Based on this H-signal model, we can obtain a canonical pair of continuously instantaneous complex envelope and phase, in which the phase consists of monotonically non-decreasing sub-components that leads to a natural definition of the instantaneous complex frequency. Moreover, the instantaneous real frequency of the complex envelope can also be well-defined if the envelope is an analytical signal.

The developed H-signal model implies an interesting extension of the multivariate signal characterization to arbitrary space dimensions, which may have potential applications in such fields where the time-frequency-amplitude information is representative for multivariate signal analysis.

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